Problem 2

Derive the formula (1.4) for the sum S_n of the geometric progression $S_n = a + ar + ar^2 + \cdots + ar^{n-1}$. *Hint:* Multiply S_n by r and subtract the result from S_n; then solve for S_n. Show that the geometric series (1.6) converges if and only if $|r| < 1$; also show that if $|r| < 1$, the sum is given by equation (1.8).

Solution

The aim here is to derive the finite summation formula (1.4),

$$S_n = a + ar + ar^2 + \cdots + ar^{n-1} = \sum_{i=0}^{n-1} ar^i = \frac{a(1-r^n)}{1-r}. \tag{1.4}$$

Multiply both sides by r.

$$rS_n = ar + ar^2 + ar^3 + \cdots + ar^n \tag{1}$$

Subtract the respective sides of equation (1.4) from those of equation (1).

$$\begin{aligned} rS_n - S_n &= (ar + ar^2 + ar^3 + \cdots + ar^n) - (a + ar + ar^2 + \cdots + ar^{n-1}) \\ &= ar^n - a \end{aligned}$$

Factor both sides.

$$(r-1)S_n = a(r^n - 1)$$

Multiply both sides by -1.

$$(1-r)S_n = a(1-r^n)$$

Divide both sides by $1-r$.

$$S_n = \frac{a(1-r^n)}{1-r}, \quad r \neq 1$$